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A BAYESIAN MODEL FOR LONGITUDINAL CIRCULAR DATA

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Abstract

The analysis of short longitudinal series of circular data may be problematic and to some extent has not been completely developed. In this paper we present a Bayesian analysis of a model for such data. The model is based on a radial projection onto the circle of a particular bivariate normal distribution. Inferences about the parameters of the model are based on samples from the corresponding joint posterior density which are obtained using a Metropolis-within-Gibbs scheme after the introduction of suitable latent variables. The procedure is illustrated both using a simulated data set and a real-data set previously analyzed in the literature.

Keywords: Circular data; longitudinal data; Gibbs sampler; latent variables; mixed-effects linear models; projected normal distribution.

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1 Introduction

Several approaches have been proposed for analyzing longitudinal data. For a review the reader is referred, for example, to Diggle *et al.* (2002), Fitzmaurice *et al.* (2004), Hedeker and Gibbons (2006), and Gelman and Hill (2007). These books all discuss longitudinal models for 'scalar' (i.e. linear) responses as opposed to circular. In contrast, methodological proposals to describe relationships within repeated measurements of circular data are rather limited. This may be due to the difficulties in working with probability distributions commonly associated with directional data and to the intrinsic dependency inherent to longitudinal structures.

Circular data are a particular case of directional data. Specifically, circular data represent directions in two dimensions. For a survey the reader is referred to Fisher *et al.* (1987), Fisher (1993), Mardia and Jupp (2000), and Jammalamadaka and SenGupta (2001). See also Arnold and SenGupta (2006) for an overview of the applications of circular data analysis in ecological and environmental sciences.

From a theoretical point of view, there are three basic approaches to directional statistics, which may be called the *embedding*, *wrapping* and *intrinsic* approaches; see, Mardia and Jupp (2000). Consequently, there are several ways of generating probability distributions for circular data. One relativaly straightforward way is to radially project on the unit circle probability distributions originally defined on the plane. In the general case, let $\boldsymbol{Y}$ be a q-dimensional random vector such that $\Pr(\boldsymbol{Y} = \boldsymbol{0}) = 0$. Then $\boldsymbol{U} = ||\boldsymbol{Y}||^{-1}\boldsymbol{Y}$ is a random point on the q-dimensional unit sphere. Its *mean direction* is the unit vector $\boldsymbol{\eta} = E(\boldsymbol{U})/\rho$, where $\rho = ||E(\boldsymbol{U})||$, $0 \leq \rho \leq 1$; here $E(\cdot)$ represents the usual expectation for random vectors, and $||\cdot||$ represents the usual Euclidean norm. The parameter ρ is called the *mean resultant length* and represents a measure of concentration for directional distributions (see, for example, Mardia and Jupp, 2000, and Presnell *et al.*, 1998).

An important instance is that in which $\boldsymbol{Y}$ has a q-variate Normal distribution, $N_q(\cdot|\boldsymbol{\mu}, \boldsymbol{\Lambda})$, with mean vector $\boldsymbol{\mu} = E(Y)$ and precision matrix $\boldsymbol{\Lambda} = \mathrm{Var}(\boldsymbol{Y})^{-1}$. In this case $\boldsymbol{U}$ is said to have a *q-dimensional projected normal distribution*, here denoted by $PN(\cdot|\boldsymbol{\mu}, \boldsymbol{\Lambda})$. In the circular case,

$q = 2$, $\boldsymbol{U}$ is a 2-dimensional unit vector, and so it can be alternatively specified by means of a single angle Θ, say. A version of the projected normal linear model for the circular case has been analyzed using a frequentist approach by Presnell *et al.* (1998). Nuñez-Antonio *et al.* (2011) present and discuss a Bayesian analysis of the same model. See also Nuñez-Antonio and Gutiérrez-Peña (2005).

There exists situations involving longitudinal relationships where the response variable is circular. However, from a theoretical point of view, there seems to be a lack of models that adequately describe longitudinal structures for circular data, and the procedures currently available to carry out inferences about these models are rather limited. In fact, there does not seem to be a general framework for the analysis of longitudinal directional data. Circular data have been studied using quasi-likelihood methods, such as the generalized estimating equations (GEE) proposed by Liang and Zeger (1986) to analyze linear data. Specifically, Arts *et al.* (2000) derive estimating equations for the parameters of a family of circular distributions with two parameters. In particular, they exhibit a case for a mixed effects model and obtain asymptotic estimates for parameters involved. In turn, Arts and Jϕrgensen (2000) have extended GEE methods to deal with Jϕrgensen's dispersion models (Jϕrgensen, 1997ab) and have applied their approach for modeling longitudinal circular data. Arts and Jϕrgensen (2000) also present a simulation study for a model which considers only the mean direction and a single covariate. They note that in some situations their proposal may have troubles with convergence, and point out that their method requires a high correlation between the longitudinal observations or large samples to achieve satisfactory performance. Recently, Song (2007) has used a generalized linear model approach where the random component belongs to the family of dispersion models. He suggests penalized pseudo-likelihood and restricted maximum likelihood estimation to bypass the analytical difficulties arising from the nonlinearity of the corresponding score functions. Nevertheless, in some cases it is not posible to get inferences for all the parameters involved in the proposed models.

Thus, previous procedures for analyzing longitudinal data for a circular response suffer from flaws that render them unfeasible to carry out inferences in general situations. These limitations include troubles for fitting, model comparison and prediction, as well as convergence problems of the iterative methods utilized, etc. The main goal of this paper is to introduce a Bayesian model to describe short series of longitudinal data where the response variable is circular. The model considers linear covariates and is based on a version of the projected bivariate normal model. In our proposal, each of the two components from model is specified by a mixed-effect linear model. In addition, we present a Bayesian analysis to build a convenient posterior distribution in order to carry out joint inferences on the all parameters in the model.

The paper is organized as follows. In the next section we introduce the projected circular longitudinal model, henceforth called the *PCL model* and describe some of its properties. In Section 3 we discuss the Bayesian analysis of the model and derive all the full conditionals needed for a Gibbs sampler. We also show how to generate samples from the corresponding joint posterior density using a Metropolis-within-Gibbs scheme. In Section 4, we present some illustrative examples. Finally, Section 5 contains some concluding remarks.

2 The PCL model

2.1 Description of the model

The aim of this work is to introduce a model to describe short series of longitudinal data, where the response is a circular variable Θ, in terms of one or more explanatory variables or covariates $\boldsymbol{x} = (x_1, ..., x_v)^t$. Even though the results presented in this study can be extended to other cases, only linear covariates will be considered here.

To introduce the PCL model we consider a multivariate perspective. Assume that measurements on each occasion j ($j = 1, \ldots, n_i$) on the ith individual in the study ($i = 1, \ldots, N$) are arranged in a $n_i \times 1$ vector of responses $\boldsymbol{\theta}_i = (\theta_{i1}, \ldots, \theta_{in_i})^t$. Thus, we have a design with N individuals and n_i angular observations, θ_{ij}, on each individual.

A first important step to construct the PCL model is to propose an augmented model via introduction of suitable set of latent variables R_{ij}, in such a way that

$$\boldsymbol{Y}_{ij} = \begin{pmatrix} Y_{ij}^{I} \\ Y_{ij}^{II} \end{pmatrix} = R_{ij} \times \begin{pmatrix} \cos\theta_{ij} \\ \sin\theta_{ij} \end{pmatrix} \quad \begin{matrix} i = 1, \cdots, N \\ j = 1, \cdots, n_i. \end{matrix}$$

where $R_{ij} = ||\boldsymbol{Y}_{ij}||$.

In addition, we propose the following structure for the vector of means

$$\boldsymbol{\mu}_{ij} = \begin{pmatrix} \mu_{ij}^{I} \\ \mu_{ij}^{II} \end{pmatrix} = \boldsymbol{B}'\boldsymbol{x}_{ij} + \boldsymbol{Z}'\boldsymbol{b}_i,$$

where

$$\begin{aligned} \mu_{ij}^{k} &= (\boldsymbol{x}_{ij}^{k})'\boldsymbol{\beta}^{k} + (\boldsymbol{z}_{ij}^{k})'\boldsymbol{b}_i^{k}, \\ &\forall \;\; k \in \{I, II\}, \\ &i = 1, \cdots, N, \text{ and} \\ &j = 1, \cdots, n_i. \end{aligned}$$

Here, $\boldsymbol{B} = [\boldsymbol{\beta}^{I}, \boldsymbol{\beta}^{II}]$ is the matrix of coefficients of model and $\boldsymbol{Z} = [\boldsymbol{b}^{I}, \boldsymbol{b}^{II}]$. Note that, in practice, each of the two components of $\boldsymbol{\mu}_{ij}$ may depend on different subsets of covariates, in which case the vectors of coefficients, $\boldsymbol{\beta}^{I}$ and $\boldsymbol{\beta}^{II}$, may have different dimensions (the same holds for vectors $\boldsymbol{b}^{I}$ y $\boldsymbol{b}^{II}$). We emphasize that, in the previous definition, we have a mixed effects model for each of the two components of the PCL model. In other words,

$$\begin{aligned} \boldsymbol{Y}_i^{I} &= \boldsymbol{X}_i^{I}\boldsymbol{\beta}^{I} + \boldsymbol{Z}_i^{I}\boldsymbol{b}_i^{I} + \boldsymbol{\varepsilon}_i^{I}, \\ \boldsymbol{Y}_i^{II} &= \boldsymbol{X}_i^{II}\boldsymbol{\beta}^{II} + \boldsymbol{Z}_i^{II}\boldsymbol{b}_i^{II} + \boldsymbol{\varepsilon}_i^{II}, \end{aligned}$$

where $\boldsymbol{Y}_i^{I}$ and $\boldsymbol{Y}_i^{II}$ are vectors of dimension n_i^k, $\forall \; i = 1, \ldots, N$. Thus, the vectors $\boldsymbol{b}_i$ represent subject-specific random effects, usually assumed to be normally distributed with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{V}$. From this perspective, it may now be realistic to assume that, given $\boldsymbol{b}_i$, the components of $\boldsymbol{\varepsilon}$ are independent. This allow us to set the precision (or covariance) matrices of ε^k as $\boldsymbol{\Lambda} = \lambda \boldsymbol{I}_{n_i}$, as in the original paper by Laird and Ware (1982). In fact, in our approach we will use the structure $\boldsymbol{\Lambda} = \boldsymbol{I}_{n_i}$ in order to appropriately arrange the construction of the PCL model.

Recalling the previous discussion, and considering each of the two components $k \in \{I, II\}$ separately, the hierarchical definition of the PCL model based on the augmented-data is

- STAGE-1. For each individual i,

  $$\boldsymbol{Y}_i^k | \, \boldsymbol{\beta}^k, \{\boldsymbol{b}_i\}^k \sim N_{n_i}(\boldsymbol{X}_i^k \boldsymbol{\beta}^k + \boldsymbol{Z}_i^k \boldsymbol{b}_i^k, \boldsymbol{I}), \qquad i = 1, \cdots, N,$$

  which means that, given $\boldsymbol{\beta}^k$ y $\{\boldsymbol{b}_i\}^k$,

  $$\boldsymbol{Y}_i^k = \boldsymbol{X}_i^k \boldsymbol{\beta}^k + \boldsymbol{Z}_i^k \boldsymbol{b}_i^k + \boldsymbol{\varepsilon}_i^k. \qquad \forall \;\; i = 1, \cdots, N,$$

  where $\boldsymbol{\varepsilon}_i^k \sim N_{n_i}(\mathbf{0}, \boldsymbol{I})$.

- STAGE-2. The vectors $\boldsymbol{\beta}^k$ and $\boldsymbol{b}_i^k$ are considered independent vectors, $\forall \;\; i = 1, \cdots, N$, with

  $$\begin{array}{lcl} \boldsymbol{b}_i^k | \boldsymbol{\Omega}^k & \sim & N_q(\mathbf{0}, \boldsymbol{\Omega}^k) \quad \forall \;\; i = 1, \cdots, N, \\ \boldsymbol{\beta}^k & \sim & N_{p_k}(\mathbf{0}, A^k). \end{array}$$

- STAGE-3.

  $$\boldsymbol{\Omega}^k \sim Wi(v^k, B^k), \quad v^k \geq q^k$$

  where q^k is dimension of vector $\boldsymbol{b}_i^k$. In this parametrization $E(\boldsymbol{\Omega}^k) = v^k (B^k)^{-1}$

2.2 Longitudinal structures obtained from the PCL model

The PCL model is flexible enough to describe a variety of longitudinal patterns for short series. Figure 1 exhibits some of these behaviors. It can be seen that the PCL model is able to reproduce the structures of random intercept, random slope and random intercept-slope of a standard mixed effects model. Furthermore, the model can also produce more general dependence patterns such as that presented in the lower right panel of Figure 1.

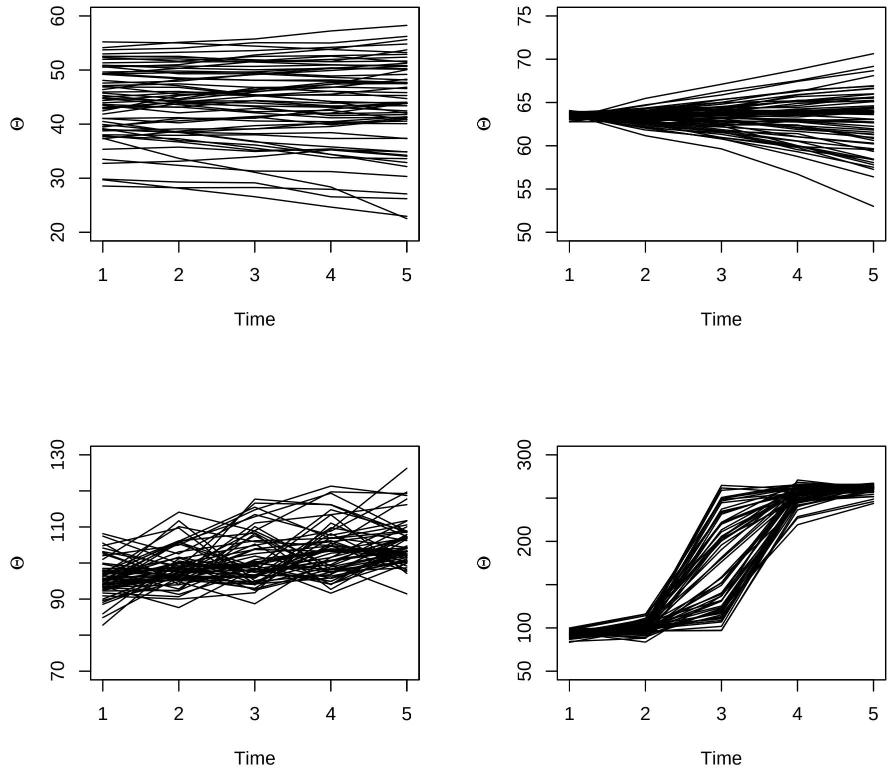

Figure 1: *Some longitudinal patterns for circular data produced by the PCL model.*

3 Inference via MCMC

Our approach is based on the introduction of suitable latent variables R_{ij} to define an augmented joint distribution for the data conditional on the matrices $\boldsymbol{B}$ y $\boldsymbol{b}$. This joint distribution is constructed in such a way as to ensure that we can simulate from all the posterior conditional densities required for a Gibbs sampler. It should be noted for each $ij-$th observation, Stage-1 of PCL model can be seen as

$$f(\boldsymbol{y}_{ij} = r_{ij}(\cos\theta, \sin\theta)^t |\ \boldsymbol{\beta}^I, \boldsymbol{\beta}^{II}, \{\boldsymbol{b}_i\}^I, \{\boldsymbol{b}_i\}^{II}, \boldsymbol{x}_{ij}^I, \boldsymbol{x}_{ij}^{II}, \boldsymbol{z}_{ij}^I, \boldsymbol{z}_{ij}^{II}) = N_2(\boldsymbol{y}_{ij}\ |\boldsymbol{\mu}_{ij}, \boldsymbol{I}).$$

In this way, if we introduce the latent variables R_{ij} defined on $(0, \infty)$ through the transformation

$$\boldsymbol{Y}_{ij} = R_{ij}(\cos\Theta_{ij}, \sin\Theta_{ij})^t,$$

then the joint density of (Θ_{ij}, R_{ij}), denoted by $f_{(\Theta_{ij},R_{ij})}(\theta_{ij}, r_{ij})$, can be obtained by letting $R_{ij} = ||\boldsymbol{Y}_{ij}||$ and then transforming to polar coordinates. It follows that Θ_{ij} has a projected normal distribution with density function given by

$$f(\theta_{ij}|\boldsymbol{\mu}_{ij}, \boldsymbol{I}) = \frac{1}{2\pi} \exp\{\frac{-1}{2}||\boldsymbol{\mu}_{ij}||^2\} \ [\ 1 \ + \frac{\boldsymbol{v}'_{ij}\boldsymbol{\mu}_{ij}}{\phi(\boldsymbol{v}^t_{ij}\boldsymbol{\mu}_{ij})}\Phi(\boldsymbol{v}'_{ij}\boldsymbol{\mu}_{ij})\ ]\ 1_{(0,2\pi]}(\theta_{ij})\ 1_{\mathbb{R}^2}(\boldsymbol{\mu}_{ij})$$

where

$$\boldsymbol{\mu}_{ij} = \begin{pmatrix} \mu^I_{ij} \\ \mu^{II}_{ij} \end{pmatrix} = \begin{pmatrix} (\boldsymbol{x}^I_{ij})^t\boldsymbol{\beta}^I + (\boldsymbol{z}^I_{ij})^t\boldsymbol{b}^I_i \\ (\boldsymbol{x}^{II}_{ij})^t\boldsymbol{\beta}^{II} + (\boldsymbol{z}^{II}_{ij})^t\boldsymbol{b}^{II}_i \end{pmatrix} \qquad \begin{array}{l} i = 1, \cdots, N, \\ j = 1, \cdots, n_i, \end{array}$$

and $\boldsymbol{v}^t_{ij} = (\cos\theta_{ij}, \sin\theta_{ij})$. Here, $\phi(\cdot)$ and $\Phi(\cdot)$ denote the probability density function and cumulative distribution function (respectively) of the standard normal distribution.

Once we have completed (augmented) the observed data, via the introduction of the latent variables R_{ij}, in order to set up the Gibbs sampler for the PCL model we must specify the corresponding conditional densities. These are described next.

3.1 Full conditional densities

Let $\boldsymbol{D_n} = \{(r_{11}, \theta_{11}), \cdots, (r_{Nn_i}, \theta_{Nn_i})\}$ be a set of observations from the PCL model. Omiting the superscript k for notational convenience, the posterior conditional densities for the paramaters and latent variables of each of the components $k \in \{I, II\}$ are given by

$$\begin{aligned} f(\boldsymbol{\beta}|\{\boldsymbol{b}_i\}, \boldsymbol{\Omega}, \boldsymbol{D_n}) &= N_p(\boldsymbol{\beta}|C^{-1}\sum_{i=1}^N \boldsymbol{X}^t_i\boldsymbol{e}_i, C). \\ f(\boldsymbol{b}_i|\boldsymbol{\beta}, \boldsymbol{\Omega}, \boldsymbol{D_n}) &= N_q(\boldsymbol{b}_i|D^{-1}_i\boldsymbol{Z}^t_i\tilde{\boldsymbol{e}}_i, D_i) \ \ \forall\ i = 1, \cdots, N. \\ f(\boldsymbol{\Omega}|\{\boldsymbol{b}_i\}, \boldsymbol{D_n}) &= Wi(v + N, B + \sum_{i=1}^N \boldsymbol{b}_i\boldsymbol{b}^t_i), \end{aligned}$$

where

$$\begin{aligned} C &= \textstyle\sum_{i=1}^{N} \boldsymbol{X}_i^t \boldsymbol{X}_i + A, \\ \boldsymbol{e}_i &= \boldsymbol{Y}_i - \boldsymbol{Z}_i \boldsymbol{b}_i, \\ D_i &= \boldsymbol{Z}_i^t \boldsymbol{Z}_i + \boldsymbol{\Omega}, \\ \tilde{\boldsymbol{e}}_i &= \boldsymbol{Y}_i - \boldsymbol{X}_i \boldsymbol{\beta}_i. \end{aligned}$$

We note that the R_{ij} are conditionally independent given the Θ_{ij}. Thus, the full conditional densities of R_{ij} are given, up to a constant of proportionality, by

$$f(r_{ij}|\{\theta_{ij}\}, \boldsymbol{\beta}, \boldsymbol{b}_i) \propto r_{ij}\ \exp\{-\frac{1}{2}\ [r_{ij}^2\ -\ 2\ (\boldsymbol{v}_{ij}^t \boldsymbol{\mu}_{ij})\ r_{ij}]\ \}.$$

Clearly, it is not difficult to sample from $f(\boldsymbol{\beta}|\{\boldsymbol{b}_i\}, \boldsymbol{\Omega}, \boldsymbol{D_n})$, $f(\boldsymbol{b}_i|\boldsymbol{\beta}, \boldsymbol{\Omega}, \boldsymbol{D_n})$ and $f(\boldsymbol{\Omega}|\{\boldsymbol{b}_i\}, \boldsymbol{D_n})$. On the other hand, we can generate R_{ij} from $f(r_{ij}|\theta_{ij}, \boldsymbol{\mu}_{ij})$ and in this way draw a random matrix $\boldsymbol{R}$ from $f(\boldsymbol{R}|\{\theta_{ij}\}, \boldsymbol{\beta}, \boldsymbol{b}_i)$. This last step is carried out via a Metropolis-Hastings algorithm.

We can now use all the previous full conditionals in a Gibbs sampler to get a sample from the joint posterior density

$$f(\boldsymbol{\beta}^I, \boldsymbol{\beta}^{II}, \{\boldsymbol{b}_i\}^I, \{\boldsymbol{b}_i\}^{II},\ \boldsymbol{\Omega}^I, \boldsymbol{\Omega}^{II}, \boldsymbol{R}|\ \{\theta_{ij}\}). \tag{3.1}$$

Unfortunately, direct implementation of the previous MCMC scheme will typically lead to a slow-mixing chain and potential convergence problems. This is due in part to the structure of the mixed effects models within each component of the projected normal distribution and to the high dimension of the hierarchical model. In the particular case of mixed effects models for longitudinal scalar data, several methods have been proposed in the literature in order to improve the efficiency of MCMC methods. See, for example, Gelfand *et al.* (1995), Vines *et al.* (1996), Gilks and Roberts (1996), Gelfand y Sahu (1999), Chib and Carlin (1999). Here, we employe a method proposed by Chib and Carlin (1999) to simulate the fixed effects $\boldsymbol{\beta}$ and all random effects $\boldsymbol{b}_i$ in a single block within each component of the PCL model.

Specifically, our algorithm to sample from the posterior distribution of all the parameters of the PCL models is the following.

- For each component $k,\ k \in \{I, II\}$,

  1. Sample $\boldsymbol{\beta}^k$ and $\{\boldsymbol{b}_i^k\}$ from $f(\boldsymbol{\beta}^k, \{\boldsymbol{b}_i\}^k | \boldsymbol{\Omega}^I, \boldsymbol{\Omega}^{II}, \boldsymbol{D_n}) = f(\boldsymbol{\beta}^k, \{\boldsymbol{b}_i\}^k | \boldsymbol{\Omega}^k, \boldsymbol{D_n})$ by sampling

     1.1 $\boldsymbol{\beta}^k$ from $f(\boldsymbol{\beta}^k | \boldsymbol{\Omega}^I, \boldsymbol{\Omega}^{II}, \boldsymbol{D_n}) = f(\boldsymbol{\beta}^k | \boldsymbol{\Omega}^k, \boldsymbol{D_n})$.

     1.2 $\boldsymbol{b}_i^k$ from $f(\boldsymbol{b}_i^k | \boldsymbol{\beta}^I, \boldsymbol{\beta}^{II}, \Omega^I, \Omega^{II}, \boldsymbol{D_n}) = f(\boldsymbol{b}_i^k | \boldsymbol{\beta}^k, \Omega^k, \boldsymbol{D_n})$,
     for each $i = 1, \ldots, N$.

  2. Sample $\boldsymbol{\Omega}^k$ from $f(\boldsymbol{\Omega}^k | \boldsymbol{\beta}^I, \boldsymbol{\beta}^{II}, \{\boldsymbol{b}_i\}^I, \{\boldsymbol{b}_i\}^{II}, \boldsymbol{D_n}) = f(\boldsymbol{\Omega}^k | \{\boldsymbol{b}_i\}^k, \boldsymbol{D_n})$.

- Sample R_{ij} from $f(r_{ij} | \boldsymbol{\beta}^I, \boldsymbol{\beta}^{II}, \{\boldsymbol{b}_i\}^I, \{\boldsymbol{b}_i\}^{II},\ \boldsymbol{\Omega}^I,\ \boldsymbol{\Omega}^{II}, \{\theta_{ij}\})$
  for each $i = 1, \ldots, N$ and each $j = 1, \ldots, n_i$.

- Repeat until convergence is achieved.

In 1.1 above, the conditional densities of $\boldsymbol{\beta}^k$, $k \in \{I, II\}$, are given by

$$f(\boldsymbol{\beta}^k | \boldsymbol{\Omega}^k, \boldsymbol{D_n}) = N_{p_k}(\boldsymbol{\beta}^k | \boldsymbol{\mu}_\beta^k, \boldsymbol{\Lambda}_{\boldsymbol{\beta}}^{\boldsymbol{k}}),$$

where

$$\begin{aligned} \boldsymbol{\mu}_\beta^k &= (\boldsymbol{\Lambda}_\beta^k)^{-1} \{ \textstyle\sum_{i=1}^N (\boldsymbol{X}_i^k)^t (\boldsymbol{V}_i^k)^{-1} \boldsymbol{Y}_i^k \}, \\ \boldsymbol{\Lambda}_\beta^k &= (A^k + \textstyle\sum_{i=1}^N (\boldsymbol{X}_i^k)^t (\boldsymbol{V}_i^k)^{-1} \boldsymbol{X}_i^k); \end{aligned}$$

see, for example, Chib and Carlin (1999).

The previous algorithm with blocking improves the mixing of the chain, and thus its convergence, especially when the blocking is applied separately to each component of the PCL model.

4 Examples

We used the R language and environment (R Development Core Team, 2011) to simulate the data set for Example 1, and to carry out all of the proposed analyses in this section.

Example 1. In this example, a longitudinal sample of size $N = 60$ was simulated. This sample represents five repeated measurements on each of $N = 60$ individuals. The data was obtained using the next specification of the PCL model:

$$\begin{aligned} \boldsymbol{Y}_i^I \mid \boldsymbol{\beta}^I, \quad &\sim \quad N_5(\boldsymbol{X}_i^I \boldsymbol{\beta}^I, \boldsymbol{I}), \\ \boldsymbol{Y}_i^{II} \mid \boldsymbol{\beta}^{II}, \{\boldsymbol{b}_i\}^{II} \quad &\sim \quad N_5(\boldsymbol{X}_i^{II} \boldsymbol{\beta}^{II} + \boldsymbol{Z}_i^{II} \boldsymbol{b}_i^{II}, \boldsymbol{I}), \qquad i = 1, \cdots, 60, \end{aligned}$$

where

$$\boldsymbol{\beta}^I = \begin{pmatrix} 100 \\ -4 \end{pmatrix}, \quad \boldsymbol{\beta}^{II} = \begin{pmatrix} 200 \\ -10 \end{pmatrix},$$

$$\boldsymbol{b}_i^{II} | \boldsymbol{\Omega}^{II} \quad \sim \quad N_2(\boldsymbol{0}, \boldsymbol{\Omega}^{II}) \qquad i = 1, \cdots, 60$$

with

$$(\boldsymbol{\Omega}^{II})^{-1} = \begin{pmatrix} 0.0001 & 0 \\ 0 & 5 \end{pmatrix},$$

and

$$X_i^I = X_i^{II} = Z_i^{II} = \begin{array}{c} \begin{array}{cc} & \text{Time} \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \end{array}, \qquad i = 1, \cdots, 60.$$

Figure 2 shows the corresponding data set. For the analysis of these data, we used a vague prior distribution with $A^I = \boldsymbol{0} = A^{II} = \boldsymbol{0}$, $v^I = v^{II} = 2$ and $B^I = B^{II} = \text{Diag}(0.001, 0.001)$.

The resulting component-wise marginal distributions for the vectors $\boldsymbol{\beta}^I$ and $\boldsymbol{\beta}^{II}$ are presented in Figure 3. Likewise, the marginal distributions for the components of the covariance matrix $(\boldsymbol{\Omega}^{II})^{-1}$ are shown in Figure 4; that is, for the elements of the matrix

$$(\boldsymbol{\Omega}^{II})^{-1} = \begin{pmatrix} (\sigma_1^2)^{II} & \sigma_{12}^{II} \\ \sigma_{12}^{II} & (\sigma_2^2)^{II} \end{pmatrix}.$$

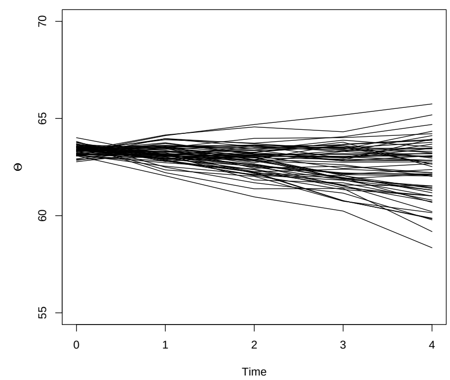

Figure 2: *Longitudinal circular data from the PCL model.*

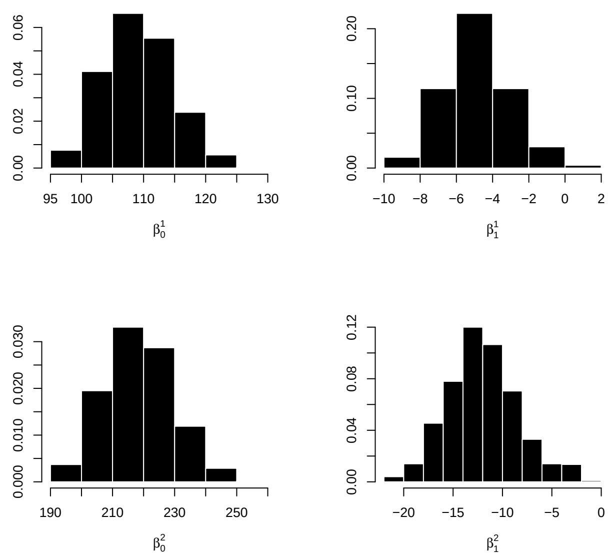

Figure 3: *Posterior densities of the elements of* $\boldsymbol{\beta}^I$ *and* $\boldsymbol{\beta}^{II}$ *for Example 1.*

In addition, the 95% posterior credible intervals for the main parameters of PCL model are presented in Table 1. It can be seen that the proposed methodology yields appropriate inferences for all parameters involved. Particularly, note that the true value of each of the parameters is well inside the highest posterior density region of the corresponding posterior density.

$\boldsymbol{\beta}^I$	$\boldsymbol{\beta}^{II}$	$(\boldsymbol{\Omega}^{II})^{-1}$
(98.872, 119.515)	(198.826, 240.078)	(0.0000, 0.445)
(-8.404 , -0.920)	(-18.399, -3.643)	(2.358, 6.318)
-	-	(-0.681, 1.519)

Table 1: *95% posterior credibility intervals for the parameters of the PCL model (Example 1).*

Example 2. For this illustration we use the proposed PCL model to analyze a real data set concerning the orientation of sandhoppers (*talitrus saltators*) escaping towards the sea in order to avoid the risk of high dehydratation. It is believed that sandhoppers will escape towards the sea, taking a course known as the *theoretical escape direction.*

Borgioli *et al.* (1999) and D'Elia *et al.* (2001) reported a longitudinal study whose aim was to help understand the escaping mechanism of sandhoppers. In this study, 65 sandhoppers were released sequentially on five occasions. Each of their escape directions was recorded, along with

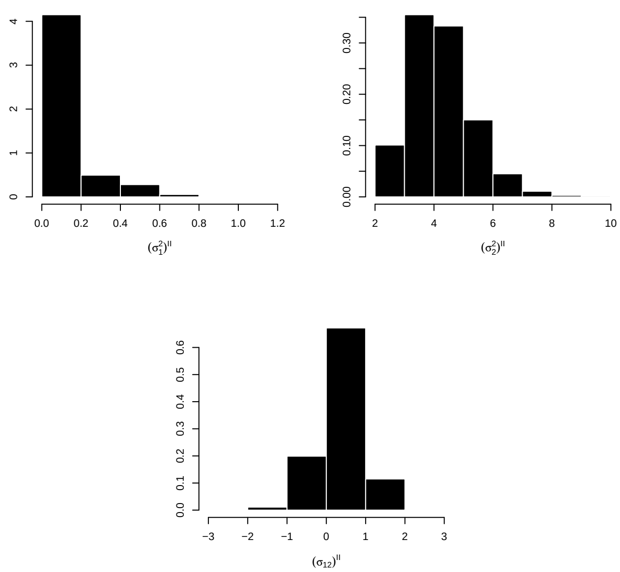

Figure 4: *Posterior densities of the elements of* $(\boldsymbol{\Omega}^{II})^{-1}$ *for Example 1.*

other covariates. The covariates included wind speed, azimuth direction for the sun (Sun), and eye measurements, which were used to construct an eye asymmetry index (Eye). The wind speeds were split into four categories (OS for offshore, LSE for longshore-east, LSW for longshore-west and Onshore), with Onshore taken as the reference category. Figure 5 shows the 65 short time series of angular responses, the escape directions.

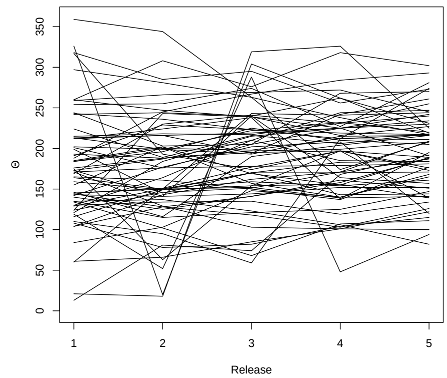

Figure 5: *Longitudinal plot of escape directions (in degrees) for sandhoppers over five consecutive releases.*

The main objective, as in Borgioli *et al.* (1999), D'Elia *et al.* (2001), and Song (2007), is to examine which covariates would significantly affect the escape direction of sandhoppers. D'Elia *et al.* (2001) and Song (2007) employed a generalized linear model approach and considered a von Mises distribution for the random component. Nevertheless, none of them offered inferences for all the parameters involved.

To analyze the sandhoppers data, here we consider a PCL model formulated as

$$\begin{aligned} \boldsymbol{\mu}_{ij}^{I} &= \beta_0^I + \beta_1^I Sun + \beta_2^I Eye + \beta_3^I OS + \beta_4^I LSW + \beta_5^I LSE + \beta_6^I Time \qquad (4.1) \\ \boldsymbol{\mu}_{ij}^{II} &= \beta_0^{II} + \beta_1^{II} Sun + \beta_2^{II} Eye + \beta_3^{II} OS + \beta_4^{II} LSW + \beta_5^{II} LSE + \beta_6^{II} Time + b_{0i} \\ & \quad i = 1, \cdots, 65. \end{aligned}$$

We used a vague prior distribution with $A^I = \mathbf{0}$, $A^{II} = \mathbf{0}$, $v^{II} = 2$ and $B^{II} = 0.001$. Figures 6 and

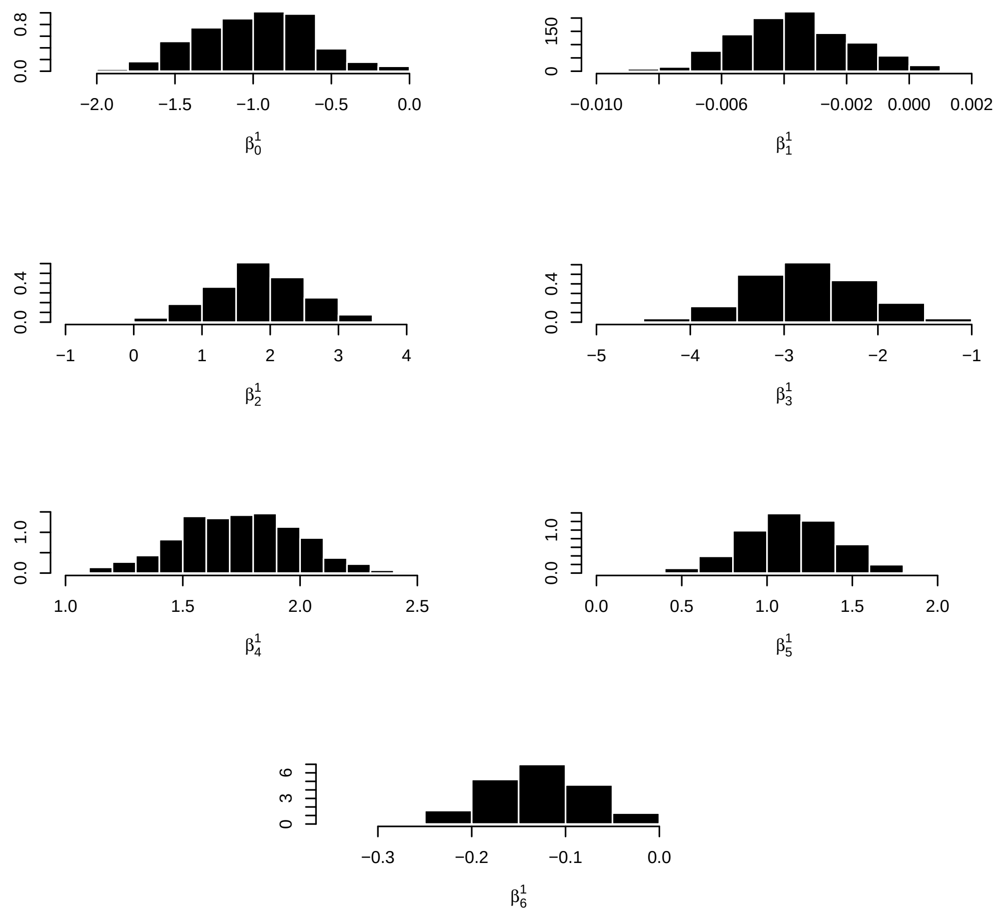

Figure 6: *Posterior distributions for the parameters of component I (sandhoppers data).*

7 show the posterior distribution of all the parameters for components I and II, respectively. In addition, Table 2 presents the corresponding 95% credibility intervals for each component of the PCL model.

This analysis suggests that $\{Sun\}$ and $\{Eye, OS, LSW\}$ are not relevant for $\boldsymbol{\mu}^I$ and $\boldsymbol{\mu}^{II}$, respectively. Moreover, the inclusion of the random effects is necessary, as the variance parameter $(\sigma^2)^{II}$ is significantly different from zero.

5 Concluding remarks

In this paper, we have introduced the PCL model, based on a projected normal distribution, for analyzing short longitudinal series of circular data. Although the PCL model assumes a conditional independence structure on each of its components, it is quite flexible and can describe several distinct longitudinal patterns. It may also provide the basis for the analysis of long (time) series of circular data. Furthermore, unlike currently available analyses of models for longitudinal circular

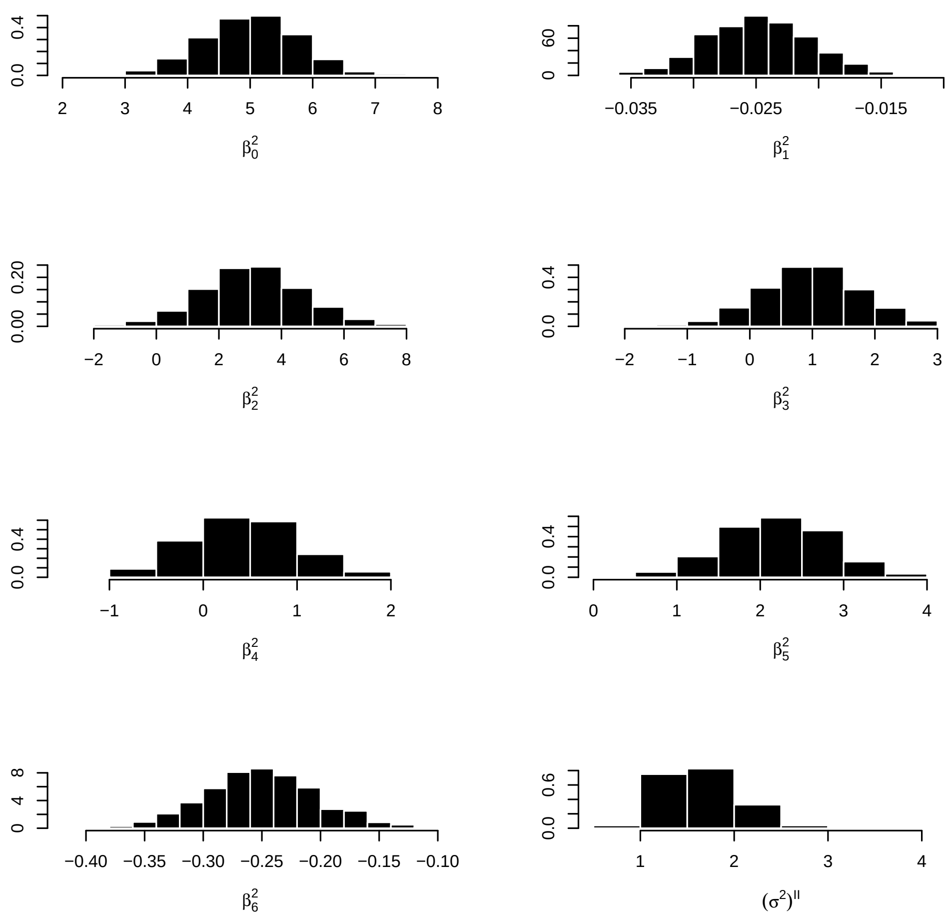

Figure 7: *Posterior distributions for the parameters of component II (sandhoppers data).*

	Component I	*Component II*
β_0	(-1.7041 , -0.2797)	(3.5300 , 6.4445)
$\beta_1(Sun)$	(-0.0069 , 0.0002)	(-0.0326 , -0.0167)
$\beta_2(Eye)$	(0.5108 , 3.1228)	(-0.2894 , 6.1439)
$\beta_3(OS)$	(-4.0097 , -1.6492)	(-0.5842 , 2.4310)
$\beta_4(LSW)$	(1.2645 , 2.2534)	(-0.6257 , 1.5565)
$\beta_5(LSE)$	(0.6042 , 1.6781)	(0.9985 , 3.4888)
$\beta_6(Time)$	(-0.2260 , -0.0277)	(-0.3420 , -0.1565)
σ^2	-	(0.9825 , 2.4505)

Table 2: 95% *credible intervals for the parameters of the PCL model (sandhoppers data).*

data, our proposal can be implemented by means of a relatively simple Gibbs sampler and can produce inferences on a variety of quantities of interest, including those related with alternative parametrizations or with prediction. An extension of this work to the analysis of time series of circular data is currently being investigated.

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